

# STATIC KLEIN-GORDON-MAXWELL-PROCA SYSTEMS IN 4-DIMENSIONAL CLOSED MANIFOLDS

EMMANUEL HEBEY AND TRONG TUONG TRUONG

ABSTRACT. We prove existence and uniform bounds for critical static Klein-Gordon-Maxwell-Proca systems in the case of 4-dimensional closed Riemannian manifolds.

Static Klein-Gordon-Maxwell-Proca systems are massive versions of the electrostatic Klein-Gordon-Maxwell Systems. The vector field in these systems inherits a mass and is governed by the Proca action which generalizes that of Maxwell. Klein-Gordon-Maxwell systems are intended to provide a dualistic model for the description of the interaction between a charged relativistic matter scalar field and the electromagnetic field that it generates. The electromagnetic field is both generated by and drives the particle field. In the electrostatic form of the Klein-Gordon-Maxwell systems, looking for standing waves  $ue^{i\omega t}$ , the matter field is characterized by the property that  $u$ , together with a gauge potential  $v$ , solve the electrostatic Klein-Gordon-Maxwell systems (0.3) with  $m_1 = 0$ . In the case of a closed manifold we discuss here the two equations in (0.3) are independent one of another when  $m_1 = 0$  and the system reduces to the sole Schrödinger equation. The Proca formalism, for  $m_1 > 0$ , leads to a deeper phenomenon and is more appropriate to the closed case. The particle in this model interacts via the minimum coupling rule

$$\partial_t \rightarrow \partial_t + iq\varphi \text{ and } \nabla \rightarrow \nabla - iqA \quad (0.1)$$

with an external massive vector field  $(\varphi, A)$  which is governed by the Maxwell-Proca Lagrangian. The Proca action is a gauge-fixed version of the Stueckelberg action in the Higgs mechanism (see Goldhaber and Nieto [26], and Ruegg and Ruiz-Altaba [43]). In the Proca formalism, developed under the influence of de Broglie, the photon inherits a nonzero mass. This issue is of considerable importance and intensively studied in modern physics (see for instance Adelberger, Dvali and Gruzinov [1], Byrne [13], Goldhaber and Nieto [25, 26], Luo and Tu [38], Luo, Gillies and Tu [37] and the references in these papers). When  $n = 3$ , the KGMP equations consist in the nonlinear Klein-Gordon matter equation, the charge continuity equation and the massive modified Maxwell equations in SI units, which are hereafter explicitly written down:

$$\begin{aligned} \nabla \cdot E &= \rho/\varepsilon_0 - \mu^2\varphi, \\ \nabla \times H &= \mu_0 \left( J + \varepsilon_0 \frac{\partial E}{\partial t} \right) - \mu^2 A, \\ \nabla \times E + \frac{\partial H}{\partial t} &= 0 \text{ and } \nabla \cdot H = 0. \end{aligned} \quad (0.2)$$

These massive Maxwell equations, as modified to Proca form, appear to have been first written in modern format by Schrödinger [46]. The Proca formalism a priori breaks Gauge invariance. Gauge invariance can be restored by the Stueckelberg trick, as pointed out by Pauli [41], and then by the Higgs mechanism. We refer to Goldhaber and Nieto [26], Luo, Gillies and Tu [37], and Ruegg and Ruiz-Altaba [43] for very complete references on the Proca approach.

In what follows we let  $(M, g)$  be a smooth compact 3, 4-dimensional Riemannian manifold. We let also  $2^* = \frac{2n}{n-2}$  be the critical Sobolev exponent, where  $n$  is the dimension of  $M$ . Given real numbers  $q > 0$ ,  $m_0, m_1 > 0$ ,  $\omega \in (-m_0, m_0)$ , and  $p \in (2, 2^*]$ , the derivation of the Klein-Gordon-Maxwell-Proca system we investigate in this paper is written as

$$\begin{cases} \Delta_g u + m_0^2 u = u^{p-1} + \omega^2 (qv - 1)^2 u \\ \Delta_g v + (m_1^2 + q^2 u^2) v = qu^2, \end{cases} \quad (0.3)$$

where  $\Delta_g = -\operatorname{div}_g \nabla$  is the Laplace-Beltrami operator. The system (0.3) corresponds to looking for standing waves  $ue^{i\omega t}$  for the full KGMP system in the static case where the massive vector field  $(\varphi, A)$  depends on the sole spatial variable. The system is energy critical when  $n = 3$  and  $p = 6$  and when  $n = 4$  and  $p = 4$ . It is subcritical otherwise, namely when  $n = 3$  and  $p \in (2, 6)$  or  $n = 4$  and  $p \in (2, 4)$ . In the above model,  $m_1$  is a coupling constant which makes that the two equations in (0.3) are truly coupled ( $m_1$  is the Proca mass in the Maxwell-Proca formalism) while  $m_0$  is the mass of the particle,  $q$  is the charge of the particle,  $u$  is the amplitude in the writing of the particle,  $\omega$  is its temporal frequency (referred to as the phase in the sequel), and  $v$  is the electric potential.

Let  $S_g$  stand for the scalar curvature of  $g$ , and  $\mathcal{S}_p(\omega)$  be the set consisting of the positive smooth solutions  $\mathcal{U} = (u, v)$  of (0.3) with phase  $\omega$  and nonlinear term  $u^{p-1}$ . Namely,

$$\mathcal{S}_p(\omega) = \left\{ (u, v) \text{ smooth s.t. } u > 0, v > 0, \text{ and } (u, v) \text{ solve (0.3)} \right\}. \quad (0.4)$$

Given  $\omega \in [0, m_0)$ , we let

$$K_0(\omega) = (-m_0, -\omega] \bigcup [\omega, m_0). \quad (0.5)$$

When  $\omega = 0$ ,  $K_0(0) = (-m_0, m_0)$  is the full admissible phase range. For  $\theta \in (0, 1)$ , and  $\mathcal{U} = (u, v)$ , we let  $\|\mathcal{U}\|_{C^{2,\theta}} = \|u\|_{C^{2,\theta}} + \|v\|_{C^{2,\theta}}$ . By a MPT solution we mean a solution with a strong mountain pass type structure. The following result was proved in Druet and Hebey [20].

**Theorem 0.1** (The 3-dimensional case - Druet and Hebey [20]). *Let  $(M, g)$  be a smooth compact 3-dimensional Riemannian manifold  $m_0, m_1 > 0$ ,  $\omega \in (-m_0, m_0)$ , and  $p \in (2, 6]$ . When  $p = 6$  assume*

$$m_0^2 < \omega^2 + \frac{1}{8} S_g(x) \quad (0.6)$$

*for all  $x \in M$ . Then (0.3) possesses a smooth positive MPT solution. Moreover, for any  $p \in (2, 6)$ , and any  $\theta \in (0, 1)$ , there exists  $C > 0$  such that for any  $\omega' \in K_0(0)$ , and any  $\mathcal{U} \in \mathcal{S}_p(\omega')$ ,  $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$ , where  $\mathcal{S}_p(\omega')$  is as in (0.4) and  $K_0(0)$  is as in (0.5). Assuming again (0.6), there also holds that for any  $\theta \in (0, 1)$ ,  $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$  for all  $\mathcal{U} \in \mathcal{S}_6(\omega')$  and all  $\omega' \in K_0(\omega)$ , where  $C > 0$  does not depend on  $\omega'$  and  $\mathcal{U}$ .*

This result exhibits phase compensation in the 3-dimensional case. We aim in this paper in proving that a similar phenomenon holds true when  $n = 4$ . In this dimension the second equation in (0.3) becomes critical and this leads to serious difficulties. We prove below the existence of smooth positive MPT solutions and the existence of uniform bounds for (0.3) in the subcritical cases  $p \in (2, 4)$  without any conditions, and in the critical case  $p = 4$  assuming that the mass potential, balanced by the phase, is smaller than the geometric threshold potential of the conformal Laplacian. In doing so we prove that phase compensation still holds true for our systems when  $n = 4$ . Our result, in the subcritical case, is as follows.

**Theorem 0.2** (The subcritical 4-dimensional case). *Let  $(M, g)$  be a smooth compact 4-dimensional Riemannian manifold,  $q > 0$ ,  $m_0, m_1 > 0$ ,  $\omega \in (-m_0, m_0)$ , and  $p \in (2, 4)$ . Then (0.3) possesses a smooth positive MPT solution. Moreover, for any  $\theta \in (0, 1)$ , there exists  $C > 0$  such that for any  $\omega' \in K_0(0)$ , and any  $\mathcal{U} \in \mathcal{S}_p(\omega')$ ,  $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$ , where  $\mathcal{S}_p(\omega')$  is as in (0.4) and  $K_0(0)$  is as in (0.5).*

In the critical case we prove the following result. The geometry of the ambient inhomogeneous space, through the scalar curvature of  $g$ , comes to play a role as in the 3-dimensional case. However, the result now turns out to be local in its existence part.

**Theorem 0.3** (The critical 4-dimensional case). *Let  $(M, g)$  be a smooth compact 4-dimensional Riemannian manifold,  $q > 0$ ,  $m_0, m_1 > 0$ ,  $\omega \in (-m_0, m_0)$ , and  $p = 4$ . Assume*

$$m_0^2 < \omega^2 + \frac{1}{6}S_g(x) \quad (0.7)$$

*for some  $x \in M$ . Then (0.3) possesses a smooth positive MPT solution. Assuming that (0.7) holds true for all  $x \in M$  there also holds that for any  $\theta \in (0, 1)$ ,  $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$  for all  $\mathcal{U} \in \mathcal{S}_4(\omega')$  and all  $\omega' \in K_0(\omega)$ , where  $C > 0$  does not depend on  $\omega'$  and  $\mathcal{U}$ ,  $\mathcal{S}_4(\omega')$  is as in (0.4), and  $K_0(\omega)$  is as in (0.5).*

There are two consequences to Theorem 0.3. We list them in points (i)-(ii) below. In point (i) we illustrate the phase compensation effect associated with (0.3). There we always get existence and a priori bounds for all phases  $\omega$  which are close to  $m_0$ . Point (ii) concerns the full range of phases when we assume  $m_0$  is not too large.

(i) Phase compensation in the critical case - Assume  $p = 4$  and  $S_g > 0$  in  $M$ . Then there exists  $\varepsilon > 0$  such that for any  $m_0 - \varepsilon < |\omega| < m_0$ , (0.3) possesses a smooth positive MPT solution. Moreover, for any  $\theta \in (0, 1)$ , there exists  $C > 0$  such that  $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$  for all  $\mathcal{U} \in \mathcal{S}_4(\omega)$  and all  $m_0 - \varepsilon < |\omega| < m_0$ .

(iii) Full phase range in the critical case - Assume  $p = 4$  and  $m_0^2 < \frac{1}{6}S_g$  in  $M$ . For any  $\omega \in (-m_0, m_0)$ , (0.3) possesses a smooth positive MPT solution. Moreover, for any  $\theta \in (0, 1)$ , there exists  $C > 0$  such that  $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$  for all  $\mathcal{U} \in \mathcal{S}_4(\omega)$  and all  $\omega \in (-m_0, m_0)$ .

As an immediate consequence of the  $C^{2,\theta}$ -bounds in the above results we obtain phase stability for standing waves of the Klein-Gordon-Maxwell-Proca equations in electrostatic form. Standing waves for the Klein-Gordon-Maxwell-Proca equations in electrostatic form are written as  $S = ue^{i\omega t}$  and they are coupled with a gauge potential  $v$ , where  $(u, v)$  solves (0.3). Roughly speaking, phase stability means that for any arbitrary sequence of standing waves  $u_\alpha e^{i\omega_\alpha t}$ , with gauge potentials  $v_\alpha$ , the convergence of the phases  $\omega_\alpha$  in  $\mathbb{R}$  implies the convergence of the amplitudes  $u_\alpha$

and of the gauge potentials  $v_\alpha$  in the  $C^2$ -topology. Phase stability prevents the existence of arbitrarily large amplitude standing waves.

High dimensional KGM systems in Coulomb gauge have been recently investigated by Rodnianski and Tao [42] and with special emphasis in  $(1+4)$ -dimensions by Klainerman and Tataru [29] and Selberg [47]. Electrostatic KGM systems in the three dimensional case have been investigated by several authors. Possible references on the physics side are by Benci and Fortunato [5, 6], Long [35], Long and Stuart [36]. Blowing-up solutions to the electrostatic Schrödinger-Maxwell system, a cousin of the electrostatic KGM type systems that we consider here, have been constructed in D'Aprile and Wei [2, 3].

We briefly discuss in Section 1 the physics relevance of (0.3). We prove our theorem in Sections 2 to 4. The existence part in the theorem is proved in Section 2. The  $C^{2,\theta}$ -bound in the subcritical case is established in Section 3. The more delicate  $C^{2,\theta}$ -bound in the critical case is established in Sections 4. The phase compensation phenomenon in the theorem holds true thanks to the 4-dimensional log effect  $\mu^2 = o(\mu^2 \log \mu)$  as  $\mu \rightarrow 0$ .

## 1. THE PHYSICS ORIGIN OF THE SYSTEM

The Klein-Gordon-Maxwell-Proca system discussed in this work describes an interacting field theory model in theoretical physics. Most electromagnetic phenomena are described by conventional electrodynamics, which is a theory of the coupling of electromagnetic fields to matter fields. Of prime importance for particle physics is fermion electrodynamics in which matter is represented by spinor fields. However one may have also boson electrodynamics in which matter is described by integer spin or bosonic fields. The simplest one is of course the complex scalar field, describing spinless particles having electric charges  $\pm q$ . It gives rise to scalar electrodynamics, which describes in the non-relativistic limit the superconductivity of metals at very low temperatures. In the more general context of particle physics, a complex scalar field  $\psi$  may serve to describe scalar mesons in nuclear matter interacting via a massive vector boson field  $(\varphi, A)$ .

The interaction in this model is described by the minimum substitution rule (0.1) in a nonlinear Klein-Gordon Lagrangian. As for the external massive vector field it is governed by the Maxwell-Proca Lagrangian. More precisely, assuming for short that the manifold is orientable, we define the Lagrangian densities  $\mathcal{L}_{NKG}$  and  $\mathcal{L}_{MP}$  of  $\psi$ ,  $\varphi$ , and  $A$  by

$$\begin{aligned}\mathcal{L}_{NKG}(\psi, \varphi, A) &= \frac{1}{2} \left| \left( \frac{\partial}{\partial t} + iq\varphi \right) \psi \right|^2 - \frac{1}{2} |(\nabla - iqA)\psi|^2 + \frac{m_0^2}{2} |\psi|^2 - \frac{1}{p} |\psi|^p, \\ \mathcal{L}_{MP}(\varphi, A) &= \frac{1}{2} \left| \frac{\partial A}{\partial t} + \nabla \varphi \right|^2 - \frac{1}{2} |\nabla \times A|^2 + \frac{m_1^2}{2} |\varphi|^2 - \frac{m_1^2}{2} |A|^2,\end{aligned}\tag{1.1}$$

where  $\nabla \times = \star d$ ,  $\star$  is the Hodge dual,  $\psi$  represents the matter complex scalar field,  $m_0$  its mass,  $q$  its charge,  $(\varphi, A)$  the electromagnetic vector field, and  $m_1$  its mass. It can be noted that  $\|(\varphi, A)\|_L^2 = |\varphi|^2 - |A|^2$  is the square of the Lorentz norm of  $(\varphi, A)$  with respect to the Lorentz metric  $\text{diag}(1, -1, \dots, -1)$ . The total action functional for  $\psi$ ,  $\varphi$ , and  $A$  is then given by

$$\mathcal{S}(\psi, \varphi, A) = \int \int (\mathcal{L}_{NKG} + \mathcal{L}_{MP}) dv_g dt. \tag{1.2}$$

Writing  $\psi$  in polar form as  $\psi(x, t) = u(x, t)e^{iS(x, t)}$ , taking the variation of  $S$  with respect to  $u$ ,  $S$ ,  $\varphi$ , and  $A$ , we get four equations which are written as

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \Delta_g u + m_0^2 u = u^{p-1} + \left( \left( \frac{\partial S}{\partial t} + q\varphi \right)^2 - |\nabla S - qA|^2 \right) u \\ \frac{\partial}{\partial t} \left( \left( \frac{\partial S}{\partial t} + q\varphi \right) u^2 \right) - \nabla \cdot \left( (\nabla S - qA) u^2 \right) = 0 \\ -\nabla \cdot \left( \frac{\partial A}{\partial t} + \nabla \varphi \right) + m_1^2 \varphi + q \left( \frac{\partial S}{\partial t} + q\varphi \right) u^2 = 0 \\ \overline{\Delta}_g A + \frac{\partial}{\partial t} \left( \frac{\partial A}{\partial t} + \nabla \varphi \right) + m_1^2 A = q (\nabla S - qA) u^2, \end{cases} \quad (1.3)$$

where  $\Delta_g = -\operatorname{div}_g \nabla$  is the Laplace-Beltrami operator,  $\overline{\Delta}_g = \delta d$  is half the Laplacian acting on forms, and  $\delta$  is the codifferential. We refer to this system as a nonlinear Klein-Gordon-Maxwell-Proca system. When  $n = 3$ ,  $\overline{\Delta}_g A = \nabla \times (\nabla \times A)$  and if we let

$$\begin{aligned} E &= - \left( \frac{\partial A}{\partial t} + \nabla \varphi \right), \quad H = \nabla \times A, \\ \rho &= - \left( \frac{\partial S}{\partial t} + q\varphi \right) qu^2, \quad \text{and } j = (\nabla S - qA) qu^2, \end{aligned} \quad (1.4)$$

then the two last equations in (1.3) give rise to the first pair of the Maxwell-Proca equations (0.2) with  $\epsilon_0 = \mu_0 = 1$  (units are chosen such that  $c = 1$ ) and  $\mu^2 = m_1^2$ , while the two first equations in (1.4) give rise to the second pair of the equations. The first equation in (1.3) gives rise to the nonlinear Klein-Gordon matter equation. The second equation in (1.3) gives rise to the charge continuity equation  $\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$  which, thanks to (0.2), is equivalent to the Lorentz condition  $\nabla \cdot A + \frac{\partial \varphi}{\partial t} = 0$ .

We assume in what follows that  $u(x, t) = u(x)$  does not depend on  $t$ ,  $S(x, t) = \omega t$  does not depend on  $x$ , and  $\varphi(x, t) = \varphi(x)$ ,  $A(x, t) = A(x)$  do not depend on  $t$ . In other words, we look for standing waves solutions of (1.3) and assume that we are in the static case of the system where  $(\varphi, A)$  depends on the sole spatial variable. By the fourth equation in (1.3) we then get that

$$\overline{\Delta}_g A + (q^2 u^2 + m_1^2) A = 0.$$

This clearly implies that, and is equivalent to,  $A \equiv 0$  since

$$\int (\overline{\Delta}_g A, A) = \int |dA|^2.$$

As a remark, assuming that  $A \equiv 0$ , the Lorentz condition for the external Proca field  $(\varphi, A)$  would make  $\varphi$  dependent on the sole spatial variables. As for the second equation in (1.3) it reduces to  $\frac{\partial^2 S}{\partial t^2} = 0$ . It is automatically satisfied when  $S(t) = \omega t$ , and we are thus left with the first and third equations in (1.3). Letting  $S = -\omega t$ , and  $\varphi = \omega v$ , these equations are rewritten as

$$\begin{cases} \Delta_g u + m_0^2 u = u^{p-1} + (q\varphi - \omega)^2 u \\ \Delta_g \varphi + m_1^2 \varphi + q(q\varphi - \omega) u^2 = 0. \end{cases} \quad (1.5)$$

In particular, letting  $\varphi = \omega v$ , in (1.5), we recover our original system (0.3). In other words, our original system (0.3) corresponds to looking for standing waves solutions of the Klein-Gordon-Maxwell-Proca system (1.3) in static form.

## 2. EXISTENCE THEORY

We prove the existence part in Theorems 0.2 and 0.3 and look for solutions with a special variational structure. Formally, solutions of (0.3) are critical points of the functional  $S$  defined by

$$\begin{aligned} S(u, v) = & \frac{1}{2} \int_M |\nabla u|^2 dv_g - \frac{\omega^2}{2} \int_M |\nabla v|^2 dv_g + \frac{m_0^2}{2} \int_M u^2 dv_g \\ & - \frac{\omega^2 m_1^2}{2} \int_M v^2 dv_g - \frac{1}{p} \int_M u^p dv_g - \frac{\omega^2}{2} \int_M u^2 (1 - qv)^2 dv_g . \end{aligned} \quad (2.1)$$

The functional  $S$  is strongly indefinite because of the competition between  $u$  and  $v$ . Following a very nice idea going back to Benci-Fortunato [5], we introduce the auxiliary functional  $\Phi$  given by

$$\Delta_g \Phi(u) + (m_1^2 + q^2 u^2) \Phi(u) = qu^2 , \quad (2.2)$$

and then consider that  $u$  in (0.3) can be seen as a critical point of

$$\begin{aligned} I_p(u) = & \frac{1}{2} \int_M |\nabla u|^2 dv_g + \frac{m_0^2}{2} \int_M u^2 dv_g - \frac{1}{p} \int_M (u^+)^p dv_g \\ & - \frac{\omega^2}{2} \int_M (1 - q\Phi(u)) u^2 dv_g , \end{aligned} \quad (2.3)$$

where  $u^+ = \max(u, 0)$ . We explicitly define MPT solutions to be solutions we obtain from  $I_p$  by the mountain pass lemma from 0 to  $u_1$  with  $\|u_1\|_{L^p}^p$  being as large as we want with respect to  $\|u_1\|_{H^1}^2$ . Let  $\Psi : H^1(M) \rightarrow \mathbb{R}$  be defined by

$$\Psi(u) = \frac{1}{2} \int_M (1 - q\Phi(u)) u^2 dv_g . \quad (2.4)$$

The following lemma establishes the existence and differentiability of  $\Phi$ , as well as the  $C^1$ -smoothness of  $\Psi$ . Equation (2.2) is critical when  $n = 4$  because of the term  $u^2 \Phi(u)$ .

**Lemma 2.1.** *Let  $(M, g)$  be a smooth compact Riemannian 4-manifold and  $q > 0$ . There exists  $\Phi : H^1(M) \rightarrow H^1(M)$  such that (2.2) holds true and  $0 \leq \Phi(u) \leq \frac{1}{q}$  for all  $u \in H^1(M)$ . Moreover,  $\Phi$  is locally Lipschitz and differentiable. Its differential  $D\Phi(u) = V_u$  at  $u$  is given by*

$$\Delta_g V_u(\varphi) + (m_1^2 + q^2 u^2) V_u(\varphi) = 2qu(1 - q\Phi(u)) \varphi \quad (2.5)$$

for all  $\varphi \in H^1(M)$ . The functional  $\Psi : H^1(M) \rightarrow \mathbb{R}$  defined in (2.4) is  $C^1$  in  $H^1(M)$  and

$$D\Psi(u) \cdot (\varphi) = \int_M (1 - q\Phi(u))^2 u \varphi dv_g \quad (2.6)$$

for all  $u, \varphi \in H^1(M)$ .

*Proof of Lemma 2.1.* We briefly sketch the proof. Let  $u \in H^1$  and  $H_u : H^1 \rightarrow \mathbb{R}$  be defined by

$$H_u(\varphi) = \int_M |\nabla \varphi|^2 dv_g + \int_M (m_1^2 + q^2 u^2) \varphi^2 dv_g .$$

The functional is well defined since  $H^1 \subset L^4$ . Letting  $\Phi(0) = 0$  we can assume that  $u \not\equiv 0$ . Let

$$\mu = \inf_{u \in H^1, \int u^2 \varphi = 1} H_u(\varphi) .$$

By standard minimization arguments there exists  $\varphi \in H^1(M)$  such that  $\int_M u^2 \varphi dv_g = 1$  and  $H_u(\varphi) = \mu$ . In particular,  $\mu > 0$ . Letting  $\Phi(u) = \frac{q}{\mu} \varphi$  we get that  $\Phi(u)$  solves (2.2) in  $H^1$ . It is easily seen that  $\Phi(u)$  is unique. By the maximum principle,  $\Phi(u) \geq 0$ . Noting that

$$\Delta_g \left( \frac{1}{q} - \Phi(u) \right) + (m_1^2 + q^2) u^2 \left( \frac{1}{q} - \Phi(u) \right) \geq 0$$

it also follows from the maximum principle that  $\Phi(u) \leq \frac{1}{q}$ . Now we let  $u, v \in H^1(M)$ . We have that

$$\Delta_g (\Phi(v) - \Phi(u)) + (m_1^2 + q^2) u^2 (\Phi(v) - \Phi(u)) = q(v^2 - u^2) (1 - q\Phi(v)) .$$

Multiplying the equation by  $\Phi(v) - \Phi(u)$ , integrating over  $M$ , and by the Sobolev emedding theorem, we get that

$$\|\Phi(v) - \Phi(u)\|_{H^1} \leq C (\|u\|_{H^1} + \|v\|_{H^1}) \|v - u\|_{H^1} . \quad (2.7)$$

In particular,  $\Phi$  is locally Lipschitz continuous. We can prove the existence of  $V_u(\varphi)$  in (2.5) as when proving the existence of  $\Phi(u)$ . Writing the equation satisfied by  $\Phi(u + \varphi) - \Phi(u) - V_u(\varphi)$ , multiplying the equation by  $\Phi(u + \varphi) - \Phi(u) - V_u(\varphi)$  and integrating over  $M$ , we get that

$$\|\Phi(u + \varphi) - \Phi(u) - V_u(\varphi)\|_{H^1} \leq C \|\varphi\|_{H^1} (\|\varphi\|_{H^1} + \|u\|_{H^1} \|\Phi(u + \varphi) - \Phi(u)\|_{H^1})$$

Then the differentiability of  $\Phi$  follows from the continuity of  $\Phi$ . In particular,  $\Psi$  is differentiable. By (2.2),

$$\Psi(u) = \frac{1}{2} \int_M (|\nabla \Phi(u)|^2 + m_1^2 \Phi(u)^2) dv_g + \frac{1}{2} \int_M (1 - q\Phi(u))^2 u^2 dv_g ,$$

and we also have that  $\frac{\partial H}{\partial \Phi}(u, \Phi(u)) = 0$ , where  $H(u, \Phi) = \frac{1}{2} H_u(\Phi) - q \int_M u^2 \Phi dv_g$ . Noting that

$$\Psi(u) = H(u, \Phi(u)) + \frac{1}{2} \int_M u^2 dv_g ,$$

we get that (2.6) holds true. The continuity of  $D\Psi$  can be proved directly from (2.6) and the continuity of  $\Phi$ . This ends the proof of the lemma.  $\square$

Now we prove the subcritical existence of Theorem 0.2. We proceed by applying the mountain pass lemma to the functional  $I_p$  in (2.3).

*Proof of existence in Theorem 0.2.* By Lemma 2.1,  $I_p$  is  $C^1$  in  $H^1$ . Let  $u_0 \in H^1$  such that  $u_0^+ \not\equiv 0$ . There holds  $I_p(0) = 0$  and  $I_p(tu_0) \rightarrow -\infty$  as  $t \rightarrow +\infty$  since  $p > 2$ . Since  $0 \leq \Phi(u) \leq \frac{1}{q}$  for all  $u$ , we also have that

$$\begin{aligned} I_p(u) &\geq \frac{1}{2} \left( \int_M |\nabla u|^2 dv_g + (m_0^2 - \omega^2) \int_M u^2 dv_g \right) - \frac{1}{p} \int_M |u|^p dv_g \\ &\geq C_1 \|u\|_{H^1}^2 - C_2 \|u\|_{H^1}^p \end{aligned}$$

for all  $u \in H^1$ , where  $C_1, C_2 > 0$  do not depend on  $u$ . In particular, there exist  $\delta, C > 0$  such that  $I_p(u) \geq C$  for all  $u \in H^1$  such that  $\|u\|_{H^1} = \delta$ . Let  $T_0 = T_0(u_0)$ ,  $T_0 \gg 1$ , be such that  $I_p(T_0 u_0) < 0$ , and  $c_p = c_p(u_0)$  be given by

$$c_p = \inf_{P \in \mathcal{P}} \max_{u \in P} I_p(u) , \quad (2.8)$$

where  $\mathcal{P}$  is the class of continuous paths joining 0 to  $T_0 u_0$ . According to the above we can apply the mountain pass lemma and we get the existence of a sequence

$(u_\alpha)_\alpha$  in  $H^1$  such that  $I_p(u_\alpha) \rightarrow c_p$  and  $DI_p(u_\alpha) \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Writing that  $I_p(u_\alpha) = c_p + o(1)$  and that  $DI_p(u_\alpha) \cdot (u_\alpha) = o(\|u_\alpha\|_{H^1})$ , we get by Lemma 2.1 that

$$\begin{aligned} & \frac{1}{2} \int_M (|\nabla u_\alpha|^2 + m_0^2 u_\alpha^2) dv_g \\ &= \frac{1}{p} \int_M (u_\alpha^+)^p dv_g + c_p + \frac{\omega^2}{2} \int_M (1 - q\Phi(u_\alpha)) u_\alpha^2 dv_g + o(1) \\ & \frac{1}{2} \int_M (|\nabla u_\alpha|^2 + m_0^2 u_\alpha^2) dv_g \\ &= \frac{1}{2} \int_M (u_\alpha^+)^p dv_g + \frac{\omega^2}{2} \int_M (1 - q\Phi(u_\alpha))^2 u_\alpha^2 dv_g + o(\|u_\alpha\|_{H^1}) \end{aligned} \quad (2.9)$$

for all  $\alpha$ . Writing that  $DI_p(u_\alpha) \cdot (u_\alpha^-) = o(\|u_\alpha^-\|_{H^1})$  we get that  $u_\alpha^- \rightarrow 0$  in  $H^1$  as  $\alpha \rightarrow +\infty$ . By (2.9) we then get that  $(u_\alpha)_\alpha$  is bounded in  $H^1$ . In particular, there exists  $u_p \in H^1(M)$  such that, up to passing to a subsequence,

(i)  $u_\alpha \rightharpoonup u_p$  weakly in  $H^1$ ,

(ii)  $u_\alpha \rightarrow u_p$  in  $L^p$ ,

and  $u_\alpha \rightarrow u_p$  a.e. as  $\alpha \rightarrow +\infty$ . Subtracting one equation to another in (2.9), letting  $\alpha \rightarrow +\infty$ , and since  $c_p \neq 0$ , we get that  $u_p \neq 0$ . Writing the equation satisfied by  $\Phi(u_\alpha) - \Phi(u_p)$ , multiplying the equation by  $\Phi(u_\alpha) - \Phi(u_p)$  and integrating over  $M$ , we get that

$$\Phi(u_\alpha) \rightarrow \Phi(u_p) \text{ in } H^1 \quad (2.10)$$

as  $\alpha \rightarrow +\infty$ . Now we let  $\varphi \in H^1$ . There holds  $DI_p(u_\alpha) \cdot (\varphi) = o(1)$ . Hence, by Lemma 2.1,

$$\begin{aligned} & \int_M \nabla u_\alpha \nabla \varphi dv_g + m_0^2 \int_M u_\alpha \varphi dv_g \\ &= \int_M (u_\alpha^+)^{p-1} \varphi dv_g + \omega^2 \int_M (1 - q\Phi(u_\alpha))^2 u_\alpha \varphi dv_g + o(1). \end{aligned} \quad (2.11)$$

Letting  $\alpha \rightarrow +\infty$  in (2.11) we then get by (2.10) that

$$\Delta_g u_p + m_0^2 u_p = (u_p^+)^{p-1} + \omega^2 (1 - q\Phi(u_p))^2 u_p$$

in  $H^1$ . Multiplying the equation by  $u_p^-$  and integrating over  $M$ , it follows that  $u_p^- \equiv 0$ . In particular,  $u_p \geq 0$ ,  $u_p \neq 0$ , and

$$\Delta_g u_p + m_0^2 u_p = u_p^{p-1} + \omega^2 (1 - q\Phi(u_p))^2 u_p \quad (2.12)$$

in  $H^1$ . By regularity results we get from (2.12) that  $u_p \in H^{2,s}$  for all  $s$ . Then, by regularity results,  $\Phi(u_p) \in H^{2,s}$  for all  $s$ . By the Sobolev embedding theorem, regularity theory, and the maximum principle, it follows that  $u_p, \Phi(u_p) \in C^2(M)$  and that  $u_p, \Phi(u_p) > 0$  in  $M$ . Letting  $u = u_p$  and  $v = \Phi(u_p)$ , this proves the existence part in Theorem 0.2.  $\square$

An additional information we obtain is that  $u_p$  realizes  $c_p$ . Indeed, since  $u_p \geq 0$ ,  $u_\alpha^- \rightarrow 0$  in  $H^1$ , and  $\Phi(u_\alpha) \rightarrow \Phi(u_p)$  in  $H^1$ , there holds that

$$\begin{aligned} & \int_M (u_\alpha^+)^p dv_g \rightarrow \int_M u_p^p dv_g, \text{ and} \\ & \int_M (1 - q\Phi(u_\alpha))^2 u_\alpha^2 dv_g \rightarrow \int_M (1 - q\Phi(u_p))^2 u_p^2 dv_g \end{aligned}$$



as  $\alpha \rightarrow +\infty$ . The second equation in (2.9) together with (2.12) then give that

$$\int_M |\nabla u_\alpha|^2 dv_g \rightarrow \int_M |\nabla u_p|^2 dv_g .$$

It follows that  $u_\alpha \rightarrow u_p$  in  $H^1$  as  $\alpha \rightarrow +\infty$ . By the first equation in (2.9) we then get that  $I_p(u_p) = c_p$ . In other words,  $c_p$  is realized by  $u_p$ . Now, given  $x_0 \in M$  and  $\rho_0 > 0$  small, sufficiently small, we define  $u_\varepsilon$  by

$$\begin{cases} u_\varepsilon(x) = \frac{\varepsilon}{\varepsilon^2 + r^2} - \frac{\varepsilon}{\varepsilon^2 + \rho_0^2} & \text{if } r \leq \rho_0 , \\ u_\varepsilon(x) = 0 & \text{if } r \geq \rho_0 , \end{cases} \quad (2.13)$$

where  $r = d_g(x_0, x)$ . Then, see Aubin [4], for any  $\lambda \in \mathbb{R}$ ,

$$J_\lambda(u_\varepsilon) = \frac{1}{K_4^2} \left( 1 + C \left( \frac{1}{6} S_g(x_0) - \lambda \right) \varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon) \right) , \quad (2.14)$$

where

$$J_\lambda(u_\varepsilon) = \frac{\int_M (|\nabla u_\varepsilon|^2 + \lambda u_\varepsilon^2) dv_g}{\left( \int_M u_\varepsilon^4 dv_g \right)^{1/2}} ,$$

and  $C > 0$  is independent of  $\alpha$ . Also there holds

$$\begin{aligned} \int_M u_\varepsilon^4 dv_g &= \int_{\mathbb{R}^3} \left( \frac{1}{1 + |x|^2} \right)^4 dx + o(1) , \\ \int_M |\nabla u_\varepsilon|^2 dv_g &= 8 \int_M u_\varepsilon^4 dv_g + o(1) . \end{aligned} \quad (2.15)$$

In what follows we prove the existence part of Theorem 0.3.

*Proof of existence in Theorem 0.3.* As a preliminary remark, by standard arguments such as developed in Aubin [4] and Brézis and Nirenberg [12], we just need to prove that we can chose  $u_0 \in H^1$ ,  $u_0^+ \not\equiv 0$ , such that

$$\delta_0 \leq c_p \leq \frac{1}{4K_4^4} - \delta_0 \quad (2.16)$$

for all  $p \in (4 - \varepsilon, 4)$  and some  $\varepsilon, \delta_0 > 0$ , where  $c_p = c_p(u_0)$  is as in (2.8). Now we assume that (0.7) holds true for some  $x \in M$ , in particular for  $x \in M$  where  $S_g$  is maximum. We let  $x_0 \in M$  be such that  $S_g$  is maximum at  $x_0$ , and  $(t_\varepsilon)_\varepsilon$  be any family of positive real numbers such that the  $t_\varepsilon$ 's are bounded. The first estimate we prove is that

$$\int_M \Phi(t_\varepsilon u_\varepsilon) u_\varepsilon^2 dv_g = O(\varepsilon^2) , \quad (2.17)$$

where the  $u_\varepsilon$ 's are as in (2.13). By definition,

$$\Delta_g \Phi(t_\varepsilon u_\varepsilon) + (m_1^2 + q^2) t_\varepsilon^2 u_\varepsilon^2 \Phi(t_\varepsilon u_\varepsilon) = q t_\varepsilon^2 u_\varepsilon^2 . \quad (2.18)$$

Multiplying (2.18) by  $\Phi(t_\varepsilon u_\varepsilon)$  and integrating over  $M$  we get by Hölder's inequalities that

$$\begin{aligned} \|\Phi(t_\varepsilon u_\varepsilon)\|_{H^1}^2 &= q t_\varepsilon^2 \int_M u_\varepsilon^2 \Phi(t_\varepsilon u_\varepsilon) dv_g \\ &\leq C \left( \int_M u_\varepsilon^{8/3} dv_g \right)^{3/4} \|\Phi(t_\varepsilon u_\varepsilon)\|_{L^4} \end{aligned}$$

and it follows from the Sobolev inequality that

$$\|\Phi(t_\varepsilon u_\varepsilon)\|_{H^1} \leq C \left( \int_M u_\varepsilon^{8/3} dv_g \right)^{3/4}. \quad (2.19)$$

Then, by (2.19),

$$\begin{aligned} \int_M \Phi(t_\varepsilon u_\varepsilon) u_\varepsilon^2 dv_g &\leq C \left( \int_M u_\varepsilon^{8/3} dv_g \right)^{3/4} \|\Phi(t_\varepsilon u_\varepsilon)\|_{L^4} \\ &\leq C \left( \int_M u_\varepsilon^{8/3} dv_g \right)^{3/2}. \end{aligned} \quad (2.20)$$

There holds,

$$\begin{aligned} \int_M u_\varepsilon^{8/3} dv_g &\leq \omega_3 \int_0^{\rho_0} \left( \frac{\varepsilon}{\varepsilon^2 + r^2} \right)^{8/3} r^3 dr \\ &= \omega_3 \varepsilon^{4/3} \int_0^{\rho_0/\varepsilon} \left( \frac{1}{1 + r^2} \right)^{8/3} r^3 dr \\ &= O(\varepsilon^{4/3}). \end{aligned} \quad (2.21)$$

By (2.20) and (2.21), this proves (2.17). Let  $(\varepsilon_\alpha)_\alpha$  be a sequence of positive real numbers such that  $\varepsilon_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ ,  $u_\alpha = u_{\varepsilon_\alpha}$ , and  $\mathcal{F}_4$  be the functional defined in  $H^1$  by

$$\mathcal{F}_4(u) = \frac{1}{2} \int_M |\nabla u|^2 dv_g + \frac{1}{2} (m_0^2 - \omega^2) \int_M u^2 dv_g - \frac{1}{4} \int_M |u|^4 dv_g. \quad (2.22)$$

By (2.15), there exists  $T_0 \gg 1$  such that  $I_4(T_0 u_\alpha) < 0$  for all  $\alpha \gg 1$ . There also holds that

$$\begin{aligned} \max_{0 \leq t \leq T_0} I_4(tu_\alpha) &\leq \max_{0 \leq t \leq T_0} \mathcal{F}_4(tu_\alpha) + CT_0^2 \max_{0 \leq t \leq T_0} \int_M \Phi(tu_\alpha) u_\alpha^2 dv_g \\ &\leq \frac{1}{4} J_\lambda(u_\alpha)^2 + CT_0^2 \max_{0 \leq t \leq T_0} \int_M \Phi(tu_\alpha) u_\alpha^2 dv_g \end{aligned}$$

for all  $\alpha$ , where  $\lambda = m_0^2 - \omega^2$ . By (2.14) and (2.17) we thus get that

$$\max_{0 \leq t \leq T_0} I_4(tu_\alpha) \leq \frac{1}{K_4^4} \left( 1 + C \left( \frac{1}{6} S_g(x_0) - m_0^2 + \omega^2 \right) \varepsilon_\alpha^2 \ln \varepsilon_\alpha + o(\varepsilon_\alpha^2 \ln \varepsilon_\alpha) \right),$$

where  $C > 0$  is independent of  $\alpha$ . By assumption the  $\varepsilon_\alpha^2 \ln \varepsilon_\alpha$  coefficient is positive. Let  $u_0 = u_\alpha$ , where  $\alpha \gg 1$  is sufficiently large such that

$$\max_{0 \leq t \leq T_0} I_4(tu_\alpha) \leq \frac{1}{4K_4^4} - \delta_0$$

for some  $\delta_0 > 0$ . Since  $u_0$  is now fixed, we can write that

$$\max_{0 \leq t \leq T_0} I_p(tu_0) \leq (1 + \delta_\varepsilon) \max_{0 \leq t \leq T_0} I_4(tu_0) \quad (2.23)$$

for all  $p \in (4 - \varepsilon, 4)$ , where  $\delta_\varepsilon > 0$  is such that  $\delta_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Noting that

$$\begin{aligned} I_p(u) &\geq \frac{1}{2} \int_M (|\nabla u|^2 + (m_0^2 - \omega^2) u^2) dv_g - \frac{1}{p} \int_M |u|^p dv_g, \\ &\geq C_1 \|u\|_{H^1}^2 - C_2 \|u\|_{H^1}^p \end{aligned}$$

where  $C_1, C_2 > 0$  are independent of  $u$ , there holds that there exist  $\delta_1, \delta_2 > 0$  such that  $\delta_1, \delta_2$  are as small as we want, and  $I_p(u) \geq \delta_2$  for all  $u$  such that  $\|u\|_{H^1} = \delta_1$ .

As a conclusion, there exist  $\delta_0 > 0$  and  $\varepsilon > 0$  such that (2.16) holds true for all  $p \in (4 - \varepsilon, 4)$ . This ends the proof of the existence part in Theorem 0.3.  $\square$

There are always constant solutions to (0.3). By (2.16) the MPT solutions we obtain are distinct from these constant solutions in several situations, e.g. like on  $S^1(T) \times S^3$  for  $T \gg 1$  when  $m_1^2/q \ll 1$ .

### 3. A PRIORI BOUNDS IN THE SUBCRITICAL CASE

We prove the uniform bounds in the subcritical case of Theorem 0.2. In what follows  $p \in (2, 4)$ .

*Proof of the uniform bounds in Theorem 0.2.* Let  $(\omega_\alpha)_\alpha$  be a sequence in  $(-m_0, m_0)$  such that  $\omega_\alpha \rightarrow \omega$  as  $\alpha \rightarrow +\infty$  for some  $\omega \in [-m_0, m_0]$ . Also let  $p \in (2, 4)$  and  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of (0.3) with phases  $\omega_\alpha$ . Then,

$$\begin{cases} \Delta_g u_\alpha + m_0^2 u_\alpha = u_\alpha^{p-1} + \omega_\alpha^2 (q v_\alpha - 1)^2 u_\alpha \\ \Delta_g v_\alpha + (m_1^2 + q^2 u_\alpha^2) v_\alpha = q u_\alpha^2 \end{cases} \quad (3.1)$$

for all  $\alpha$ . By the second equation in (3.1),  $0 \leq v_\alpha \leq \frac{1}{q}$  for all  $\alpha$ . Assume by contradiction that

$$\max_M u_\alpha \rightarrow +\infty \quad (3.2)$$

as  $\alpha \rightarrow +\infty$ . Let  $x_\alpha \in M$  and  $\mu_\alpha > 0$  be given by

$$u_\alpha(x_\alpha) = \max_M u_\alpha = \mu_\alpha^{-2/(p-2)}.$$

By (3.2),  $\mu_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Define  $\tilde{u}_\alpha$  by

$$\tilde{u}_\alpha(x) = \mu_\alpha^{\frac{2}{p-2}} u_\alpha(\exp_{x_\alpha}(\mu_\alpha x))$$

and  $g_\alpha$  by  $g_\alpha(x) = (\exp_{x_\alpha}^* g)(\mu_\alpha x)$  for  $x \in B_0(\delta \mu_\alpha^{-1})$ , where  $\delta > 0$  is small. Since  $\mu_\alpha \rightarrow 0$ , we get that  $g_\alpha \rightarrow \xi$  in  $C_{loc}^2(\mathbb{R}^3)$  as  $\alpha \rightarrow +\infty$ . Moreover, by (3.1),

$$\Delta_{g_\alpha} \tilde{u}_\alpha + \mu_\alpha^2 m_0^2 \tilde{u}_\alpha = \tilde{u}_\alpha^{p-1} + \omega_\alpha^2 \mu_\alpha^2 (q \hat{v}_\alpha - 1)^2 \tilde{u}_\alpha, \quad (3.3)$$

where  $\hat{v}_\alpha$  is given by  $\hat{v}_\alpha(x) = v_\alpha(\exp_{x_\alpha}(\mu_\alpha x))$ . We have  $\tilde{u}_\alpha(0) = 1$  and  $0 \leq \tilde{u}_\alpha \leq 1$ . By (3.3) and standard elliptic theory arguments, we can write that, after passing to a subsequence,  $\tilde{u}_\alpha \rightarrow u$  in  $C_{loc}^{1,\theta}(\mathbb{R}^4)$  as  $\alpha \rightarrow +\infty$ , where  $u$  is such that  $u(0) = 1$  and  $0 \leq u \leq 1$ . Then

$$\Delta u = u^{p-1}$$

in  $\mathbb{R}^4$ , where  $\Delta$  is the Euclidean Laplacian. It follows that  $u$  is actually smooth and positive, and, since  $2 < p < 4$ , we get a contradiction with the Liouville result of Gidas and Spruck [24]. As a conclusion, (3.2) is not possible and there exists  $C > 0$  such that

$$u_\alpha + v_\alpha \leq C \quad (3.4)$$

in  $M$  for all  $\alpha$ . Coming back to (3.1) it follows that the sequences  $(u_\alpha)_\alpha$  and  $(v_\alpha)_\alpha$  are actually bounded in  $H^{2,s}$  for all  $s$ . Pushing one step further the regularity argument they turn out to be bounded in  $H^{3,s}$  for all  $s$ , and by the Sobolev embedding theorem we get that they are also bounded in  $C^{2,\theta}$ ,  $0 < \theta < 1$ . This ends the proof of the uniform bounds in Theorem 0.2 when  $p \in (2, 4)$ .  $\square$

If we assume that  $\omega_\alpha \rightarrow \omega$  as  $\alpha \rightarrow +\infty$  for some  $\omega \in (-m_0, m_0)$ ,  $p \in (2, 4]$ , and  $u_\alpha \rightarrow u$  and  $v_\alpha \rightarrow v$  in  $C^2$  as  $\alpha \rightarrow +\infty$ , then  $u > 0$ ,  $v > 0$ , and  $u, v$  are smooth solutions of (0.3). Indeed, given  $\varepsilon > 0$  sufficiently small, since  $m_0^2 - \omega^2 > 0$ ,  $\Delta_g + (m_0^2 - \omega^2 - \varepsilon)$  is coercive. There holds that  $0 \leq v_\alpha \leq \frac{1}{q}$  for all  $\alpha$ . In particular, by (3.1) and the Sobolev inequality, for any  $\alpha \gg 1$  sufficiently large,

$$\begin{aligned} & \int_M (|\nabla u_\alpha|^2 + (m_0^2 - \omega^2 - \varepsilon) u_\alpha^2) dv_g \\ & \leq \int_M |\nabla u_\alpha|^2 dv_g + m_0^2 \int_M u_\alpha^2 dv_g - \omega_\alpha^2 \int_M (qv_\alpha^2 - 1)^2 u_\alpha^2 dv_g \\ & = \int_M u_\alpha^p dv_g \leq C \left( \int_M (|\nabla u_\alpha|^2 + (m_0^2 - \omega^2 - \varepsilon) u_\alpha^2) dv_g \right)^{p/2} \end{aligned}$$

for some  $C > 0$  independent of  $\alpha$ . This implies  $u > 0$  and then  $v > 0$ . Obviously the positivity of  $u$  and  $v$  does not hold anymore if we allow  $\omega^2 = m_0^2$ . Let  $(\varepsilon_\alpha)_\alpha$  be a sequence of positive real numbers such that  $\varepsilon_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Let  $u_\alpha = \varepsilon_\alpha$  and

$$v_\alpha = \frac{q\varepsilon_\alpha^2}{m_1^2 + q^2\varepsilon_\alpha^2}.$$

Then  $u_\alpha \rightarrow 0$  and  $v_\alpha \rightarrow 0$  in  $C^2$  as  $\alpha \rightarrow +\infty$ , and we do have that  $(u_\alpha, v_\alpha)$  solves (3.1), where

$$\omega_\alpha^2 = \frac{1}{(qv_\alpha - 1)^2} (m_0^2 - \varepsilon_\alpha^{p-2}).$$

In this case  $\omega_\alpha^2 \rightarrow m_0^2$  as  $\alpha \rightarrow +\infty$  and the construction provides a counter example to the above statement about the positivity of  $u$  and  $v$ .

#### 4. A PRIORI BOUNDS IN THE CRITICAL CASE

In what follows we let  $(M, g)$  be a smooth compact 4-dimensional Riemannian manifold,  $m_0, m_1 > 0$ , and  $(\omega_\alpha)_\alpha$  be a sequence in  $(-m_0, m_0)$  such that  $\omega_\alpha \rightarrow \omega$  as  $\alpha \rightarrow +\infty$  for some  $\omega \in [-m_0, m_0]$ . Also we let  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of (0.3) with phases  $\omega_\alpha$  and  $p = 4$ . Namely,

$$\begin{cases} \Delta_g u_\alpha + m_0^2 u_\alpha = u_\alpha^3 + \omega_\alpha^2 (qv_\alpha - 1)^2 u_\alpha \\ \Delta_g v_\alpha + (m_1^2 + q^2 u_\alpha^2) v_\alpha = qu_\alpha^2 \end{cases} \quad (4.1)$$

for all  $\alpha$ . By the second equation in (4.1),  $0 \leq v_\alpha \leq \frac{1}{q}$  for all  $\alpha$ . In particular, if we let

$$h_\alpha = m_0^2 - \omega_\alpha^2 (qv_\alpha - 1)^2, \quad (4.2)$$

then  $\|h_\alpha\|_{L^\infty} \leq C$  for all  $\alpha$ , where  $C > 0$  is independent of  $\alpha$ . Assume by contradiction that

$$\max_M u_\alpha \rightarrow +\infty \quad (4.3)$$

as  $\alpha \rightarrow +\infty$ . In what follows we let  $(x_\alpha)_\alpha$  be a sequence of points in  $M$ , and  $(\rho_\alpha)_\alpha$  be a sequence of positive real numbers,  $0 < \rho_\alpha < i_g/7$  for all  $\alpha$ , where  $i_g$  is the injectivity radius of  $(M, g)$ . We assume that the  $x_\alpha$ 's and  $\rho_\alpha$ 's satisfy

$$\begin{cases} \nabla u_\alpha(x_\alpha) = 0 \text{ for all } \alpha, \\ d_g(x_\alpha, x) u_\alpha(x) \leq C \text{ for all } x \in B_{x_\alpha}(7\rho_\alpha) \text{ and all } \alpha, \\ \lim_{\alpha \rightarrow +\infty} \rho_\alpha \sup_{B_{x_\alpha}(6\rho_\alpha)} u_\alpha(x) = +\infty. \end{cases} \quad (4.4)$$

We let  $\mu_\alpha$  be given by

$$\mu_\alpha = u_\alpha(x_\alpha)^{-1}. \quad (4.5)$$

Since the  $h_\alpha$ 's in (4.2) are  $L^\infty$ -bounded we can apply the asymptotic analysis in Druet and Hebey [18] and Druet, Hebey and Vétois [22]. In particular, we get that  $\frac{\rho_\alpha}{\mu_\alpha} \rightarrow +\infty$  as  $\alpha \rightarrow +\infty$  and that

$$\mu_\alpha u_\alpha(\exp_{x_\alpha}(\mu_\alpha x)) \rightarrow \left(1 + \frac{|x|^2}{8}\right)^{-1} \quad (4.6)$$

in  $C_{loc}^1(\mathbb{R}^4)$  as  $\alpha \rightarrow +\infty$ , where  $\mu_\alpha$  is as in (4.5). As a consequence,  $\mu_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Now we define  $\varphi_\alpha : (0, \rho_\alpha) \mapsto \mathbb{R}^+$  by

$$\varphi_\alpha(r) = \frac{1}{|\partial B_{x_\alpha}(r)|_g} \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g, \quad (4.7)$$

where  $|\partial B_{x_\alpha}(r)|_g$  is the volume of the sphere of center  $x_\alpha$  and radius  $r$  for the induced metric. Let  $\Lambda = 4\sqrt{2}$ . We define  $r_\alpha \in [\Lambda\mu_\alpha, \rho_\alpha]$  by

$$r_\alpha = \sup \{r \in [\Lambda\mu_\alpha, \rho_\alpha] \text{ s.t. } (s\varphi_\alpha(s))' \leq 0 \text{ in } [\Lambda\mu_\alpha, r]\}. \quad (4.8)$$

It follows from (4.6) that

$$\frac{r_\alpha}{\mu_\alpha} \rightarrow +\infty \quad (4.9)$$

as  $\alpha \rightarrow +\infty$ , while the definition of  $r_\alpha$  gives that

$$r\varphi_\alpha(r) \text{ is non-increasing in } [\Lambda\mu_\alpha, r_\alpha] \quad (4.10)$$

and that

$$(r\varphi_\alpha(r))'(r_\alpha) = 0 \text{ if } r_\alpha < \rho_\alpha. \quad (4.11)$$

Let  $B_\alpha$  be defined in  $M$  by

$$B_\alpha(x) = \frac{\mu_\alpha}{\mu_\alpha^2 + \frac{d_g(x_\alpha, x)^2}{8}}, \quad (4.12)$$

where  $\mu_\alpha$  is as in (4.5). The following sharp estimates, see Druet, Hebey and Robert [21] and Druet, Hebey and Vétois [22], hold true.

**Lemma 4.1.** *Let  $(M, g)$  be a smooth compact Riemannian 4-dimensional manifold, and  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of (4.1) such that (4.3) holds true. Let  $(x_\alpha)_\alpha$  and  $(\rho_\alpha)_\alpha$  be such that (4.4) hold true, and let  $R \geq 6$  be such that  $Rr_\alpha \leq 6\rho_\alpha$  for all  $\alpha \gg 1$ . There exists  $C > 0$  such that, after passing to a subsequence,*

$$u_\alpha(x) + d_g(x_\alpha, x) |\nabla u_\alpha(x)| \leq C\mu_\alpha d_g(x_\alpha, x)^{-2} \quad (4.13)$$

*for all  $x \in B_{x_\alpha}(\frac{R}{2}r_\alpha) \setminus \{x_\alpha\}$  and all  $\alpha$ , where  $\mu_\alpha$  is as in (4.5), and where  $r_\alpha$  is as in (4.8). In addition, there also exist  $C > 0$  and  $(\varepsilon_\alpha)_\alpha$  such that*

$$|u_\alpha - B_\alpha| \leq C\mu_\alpha (r_\alpha^{-2} + S_\alpha) + \varepsilon_\alpha B_\alpha \quad (4.14)$$

*in  $B_{x_\alpha}(2r_\alpha) \setminus \{x_\alpha\}$  for all  $\alpha$ , where  $\varepsilon_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$  and  $S_\alpha(x) = d_g(x_\alpha, x)^{-1}$  for  $x \in M \setminus \{x_\alpha\}$ .*

Lemma 4.1 provide a sharp control on the  $u_\alpha$ 's, but we need more to conclude. We prove that the following fundamental asymptotic estimate holds true. Lemma 4.2 is the key estimate we need to prove the a priori bounds in the critical case discussed in this section.

**Lemma 4.2.** *Let  $(M, g)$  be a smooth compact Riemannian 4-dimensional manifold and  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of (4.1) such that (4.3) holds true. Let  $(x_\alpha)_\alpha$  and  $(\rho_\alpha)_\alpha$  be such that (4.4) holds true. Assume (0.7). There holds that  $r_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ , where  $r_\alpha$  is as in (4.8). Moreover  $\rho_\alpha = O(r_\alpha)$  and*

$$r_\alpha^2 \mu_\alpha^{-1} u_\alpha(\exp_{x_\alpha}(r_\alpha x)) \rightarrow \frac{8}{|x|^2} + \mathcal{H}(x) \quad (4.15)$$

in  $C_{loc}^2(B_0(2) \setminus \{0\})$  as  $\alpha \rightarrow +\infty$ , where  $\mu_\alpha$  is as in (4.5), and  $\mathcal{H}$  is a harmonic function in  $B_0(2)$  which satisfies that  $\mathcal{H}(0) \leq 0$ .

*Proof of Lemma 4.2.* Let  $R \geq 6$  be such that  $Rr_\alpha \leq 6\rho_\alpha$  for  $\alpha \gg 1$ . We assume first that  $r_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . For  $x \in B_0(3)$  we define

$$\begin{aligned} \tilde{u}_\alpha(x) &= r_\alpha^2 \mu_\alpha^{-1} u_\alpha(\exp_{x_\alpha}(r_\alpha x)) , \\ \tilde{g}_\alpha(x) &= (\exp_{x_\alpha}^* g)(r_\alpha x) , \text{ and} \\ \tilde{h}_\alpha(x) &= h_\alpha(\exp_{x_\alpha}(r_\alpha x)) , \end{aligned}$$

where  $h_\alpha$  is as in (4.2). Since  $r_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ , we have that  $\tilde{g}_\alpha \rightarrow \xi$  in  $C_{loc}^2(\mathbb{R}^n)$  as  $\alpha \rightarrow +\infty$ , where  $\xi$  is the Euclidean metric. Thanks to Lemma 4.1,

$$|\tilde{u}_\alpha(x)| \leq C|x|^{-2} \quad (4.16)$$

in  $B_0(\frac{R}{2}) \setminus \{0\}$ . By (4.1),

$$\Delta_{\tilde{g}_\alpha} \tilde{u}_\alpha + r_\alpha^2 \tilde{h}_\alpha \tilde{u}_\alpha = \left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \tilde{u}_\alpha^3 \quad (4.17)$$

in  $B_0(\frac{R}{2})$ . Thanks to (4.9) and by standard elliptic theory, we then deduce that, after passing to a subsequence,

$$\tilde{u}_\alpha \rightarrow \tilde{u} \quad (4.18)$$

in  $C_{loc}^2(B_0(\frac{R}{2}) \setminus \{0\})$  as  $\alpha \rightarrow +\infty$ , where  $\mathcal{W}$  satisfies  $\Delta \tilde{u} = 0$  in  $B_0(\frac{R}{2}) \setminus \{0\}$  and  $\Delta$  is the Euclidean Laplace Beltrami operator. Moreover, thanks to (4.16), we know that

$$|\tilde{u}(x)| \leq C|x|^{-2} \quad (4.19)$$

in  $B_0(\frac{R}{2}) \setminus \{0\}$ . Thus we can write that

$$\tilde{u}(x) = \frac{\Lambda}{|x|^2} + \mathcal{H}(x) \quad (4.20)$$

where  $\Lambda \geq 0$  and  $\mathcal{H}$  satisfies  $\Delta \mathcal{H} = 0$  in  $B_0(\frac{R}{2})$ . In order to see that  $\Lambda = 8$ , it is sufficient to integrate (4.17) in  $B_0(1)$  to get that

$$-\int_{\partial B_0(1)} \partial_\nu \tilde{u}_\alpha d\sigma_{\tilde{g}_\alpha} = \left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \int_{B_0(1)} \tilde{u}_\alpha^3 dv_{\tilde{g}_\alpha} - r_\alpha^2 \int_{B_0(1)} \tilde{h}_\alpha \tilde{u}_\alpha dv_{\tilde{g}_\alpha} . \quad (4.21)$$

By (4.16),

$$\int_{B_0(1)} \tilde{u}_\alpha dv_{\tilde{g}_\alpha} \leq C \quad (4.22)$$

and by changing  $x$  into  $\frac{\mu_\alpha}{r_\alpha} x$ , we can write that

$$\int_{B_0(1)} \tilde{u}_\alpha^3 dv_{g_\alpha} = r_\alpha^2 \mu_\alpha^{-2} \int_{B_0(\frac{r_\alpha}{\mu_\alpha})} \hat{u}_\alpha^3 dv_{\hat{g}_\alpha} ,$$

where  $\hat{u}_\alpha(x) = \mu_\alpha u_\alpha(\exp_{x_\alpha}(\mu_\alpha x))$  and  $\hat{g}_\alpha(x) = (\exp_{x_\alpha}^* g)(\mu_\alpha x)$ . By (4.6) and Lemma 4.1, we then get that

$$\lim_{\alpha \rightarrow +\infty} \left( \frac{\mu_\alpha}{r_\alpha} \right)^2 \int_{B_0(1)} \tilde{u}_\alpha^3 dv_{\tilde{g}_\alpha} = 16\omega_3. \quad (4.23)$$

Noting that by (4.18) and (4.20),

$$\lim_{\alpha \rightarrow +\infty} \int_{\partial B_0(1)} \partial_\nu \tilde{u}_\alpha d\sigma_{\tilde{g}_\alpha} = -2\omega_3 \Lambda, \quad (4.24)$$

we get that  $\Lambda = 8$  thanks to (4.22)–(4.24) by passing into the limit in (4.21) as  $\alpha \rightarrow +\infty$ . At this point we claim that there exists  $\beta \in (0, 1]$  and  $C > 0$  such that

$$v_\alpha \leq C u_\alpha^\beta \text{ in } M \quad (4.25)$$

for all  $\alpha$ . Let  $x_\alpha \in M$  be a point where  $\frac{v_\alpha}{u_\alpha}$  is maximum. Then,

$$\frac{\Delta_g v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)} \geq \frac{\Delta_g u_\alpha^\beta(x_\alpha)}{u_\alpha^\beta(x_\alpha)}$$

and it follows from (4.1) that

$$\begin{aligned} & q \frac{u_\alpha(x_\alpha)^2}{v_\alpha(x_\alpha)} - m_1^2 - q^2 u_\alpha(x_\alpha)^2 \\ & \geq -\beta(\beta - 1) \frac{|\nabla u_\alpha(x_\alpha)|^2}{u_\alpha(x_\alpha)^2} + \beta u_\alpha(x_\alpha)^2 - \beta m_0^2 + \beta \omega_\alpha^2 (q v_\alpha(x_\alpha) - 1)^2. \end{aligned} \quad (4.26)$$

Choosing  $\beta \in (0, 1]$  such that  $m_1^2 - \beta m_0^2 > 0$ , since  $0 < v_\alpha \leq \frac{1}{q}$ , we get that  $u_\alpha^\beta(x_\alpha) \geq C v_\alpha(x_\alpha)$  for some  $C > 0$  independent of  $\alpha$ . This proves (4.25). In what follows we let  $X_\alpha$  be the 1-form given by

$$X_\alpha(x) = \left( 1 - \frac{1}{18} \text{Rc}_g^\sharp(x) \cdot (\nabla f_\alpha(x), \nabla f_\alpha(x)) \right) \nabla f_\alpha(x), \quad (4.27)$$

where  $f_\alpha(x) = \frac{1}{2} d_g(x_\alpha, x)^2$ ,  $\text{Rc}_g$  is the Ricci curvature of  $g$ , and  $\sharp$  is the musical isomorphism. We apply the Pohozaev identity in Druet-Hebey [19] with the vector field  $X_\alpha$  to  $u_\alpha$  in  $B_{x_\alpha}(r_\alpha)$ . We separate the regular part  $A_\alpha = m_0^2 - \omega_\alpha^2$  from the singular part in  $h_\alpha$ . Then,  $h_\alpha = A_\alpha + O(v_\alpha)$  and we get that

$$\begin{aligned} & \int_{B_{x_\alpha}(r_\alpha)} A_\alpha u_\alpha X_\alpha(\nabla u_\alpha) dv_g + \frac{1}{8} \int_{B_{x_\alpha}(r_\alpha)} (\Delta_g \text{div}_g X_\alpha) u_\alpha^2 dv_g \\ & + \frac{1}{4} \int_{B_{x_\alpha}(r_\alpha)} (\text{div}_g X_\alpha) A_\alpha u_\alpha^2 dv_g \\ & = Q_{1,\alpha} + Q_{2,\alpha} + Q_{3,\alpha} + O \left( \int_{B_{x_\alpha}(r_\alpha)} v_\alpha u_\alpha^2 dv_g \right) \\ & + O \left( \int_{B_{x_\alpha}(r_\alpha)} v_\alpha u_\alpha |X_\alpha(\nabla u_\alpha)| dv_g \right), \end{aligned} \quad (4.28)$$

where

$$\begin{aligned}
Q_{1,\alpha} &= \frac{1}{4} \int_{\partial B_{x_\alpha}(r_\alpha)} (\operatorname{div}_g X_\alpha) u_\alpha \partial_\nu u_\alpha d\sigma_g \\
&\quad - \int_{\partial B_{x_\alpha}(r_\alpha)} \left( \frac{1}{2} X_\alpha(\nu) |\nabla u_\alpha|^2 - X_\alpha(\nabla u_\alpha) \partial_\nu u_\alpha \right) d\sigma_g, \\
Q_{2,\alpha} &= - \sum_{i=1}^p \int_{B_{x_\alpha}(r_\alpha)} \left( \nabla X_\alpha - \frac{1}{4} (\operatorname{div}_g X_\alpha) g \right)^\sharp (\nabla u_\alpha, \nabla u_\alpha) dv_g, \\
Q_{3,\alpha} &= \frac{1}{4} \int_{\partial B_{x_\alpha}(r_\alpha)} X_\alpha(\nu) u_\alpha^4 d\sigma_g - \frac{1}{8} \int_{\partial B_{x_\alpha}(r_\alpha)} (\partial_\nu \operatorname{div}_g X_\alpha) u_\alpha^2 d\sigma_g,
\end{aligned}$$

and  $\nu$  is the unit outward normal derivative to  $B_{x_\alpha}(r_\alpha)$ . We have that

$$\begin{aligned}
|X_\alpha(x)| &= O(d_g(x_\alpha, x)) \quad , \quad \operatorname{div}_g X_\alpha(x) = n + O(d_g(x_\alpha, x)^2) \quad , \\
|\nabla(\operatorname{div}_g X_\alpha)(x)| &= O(d_g(x_\alpha, x)) \quad , \\
\text{and } \Delta_g(\operatorname{div}_g X_\alpha)(x) &= \frac{4}{3} S_g(x_\alpha) + O(d_g(x_\alpha, x)) \quad .
\end{aligned} \tag{4.29}$$

Following Druet, Hebey and Vétois [22] we get from Lemma 4.1, (4.28) and (4.29) that

$$\begin{aligned}
Q_{1,\alpha} &= -64\omega_3 \left( m_0^2 - \omega^2 - \frac{1}{6} S_g(x_0) \right) \mu_\alpha^2 \ln \frac{r_\alpha}{\mu_\alpha} \\
&\quad + o \left( \mu_\alpha^2 \ln \frac{1}{\mu_\alpha} \right) + o(\mu_\alpha^2 r_\alpha^{-2}) + O \left( \int_{B_{x_\alpha}(r_\alpha)} v_\alpha u_\alpha^2 dv_g \right) \\
&\quad + O \left( \int_{B_{x_\alpha}(r_\alpha)} v_\alpha u_\alpha |X_\alpha(\nabla u_\alpha)| dv_g \right) ,
\end{aligned} \tag{4.30}$$

where  $x_\alpha \rightarrow x_0$  as  $\alpha \rightarrow +\infty$ . By Lemma 4.1 and (4.29) there also holds that

$$Q_{1,\alpha} = O(\mu_\alpha^2 r_\alpha^{-2}) . \tag{4.31}$$

At this point we decompose  $v_\alpha$  into a quasi-harmonic part with nonzero Dirichlet boundary condition and a quasi-Poisson part with zero Dirichlet boundary condition. More precisely, we write that

$$v_\alpha = w_{1,\alpha} + w_{2,\alpha} \tag{4.32}$$

in  $B_\alpha = B_{x_\alpha}(\hat{r}_\alpha)$ , where  $\hat{r}_\alpha = \frac{5}{2}r_\alpha$ , and  $w_{1,\alpha}, w_{2,\alpha}$  are given by

$$\begin{cases} \Delta_g w_{1,\alpha} + m_1^2 w_{1,\alpha} = 0 & \text{in } B_\alpha \\ w_{1,\alpha} = v_\alpha & \text{on } \partial B_\alpha , \end{cases} \tag{4.33}$$

and if  $W_\alpha = \Delta_g v_\alpha + m_1^2 v_\alpha$ , by

$$\begin{cases} \Delta_g w_{2,\alpha} + m_1^2 w_{2,\alpha} = W_\alpha & \text{in } B_\alpha \\ w_{2,\alpha} = 0 & \text{on } \partial B_\alpha . \end{cases} \tag{4.34}$$

Let  $G_\alpha$  be the Green's function of  $\Delta_g + m_1^2$  in  $B_\alpha$  with zero Dirichlet boundary condition on  $\partial B_\alpha$ . By the maximum principle, considering the Green's function on a larger ball of radius  $i_g$ , we obtain by comparison of the two Green's functions that



there exists  $C > 0$  such that  $G_\alpha(x, y) \leq C d_g(x, y)^{-2}$  for all  $x \neq y$  in  $B_\alpha$ . Writing that

$$w_{2,\alpha}(x) = \int_{B_\alpha} G_\alpha(x, y) W_\alpha(y) dv_g(y)$$

it follows that

$$|w_{2,\alpha}(x)| \leq C \int_{B_\alpha} \frac{u_\alpha^2(y) dv_g(y)}{d_g(x, y)^2}. \quad (4.35)$$

By (4.6) and Lemma 4.1 we can write that

$$u_\alpha(x) \leq \frac{C \mu_\alpha}{\mu_\alpha^2 + d_g(x_\alpha, x)^2} \quad (4.36)$$

in  $B_\alpha$ . Combining (4.35) and (4.36) we then get that

$$|w_{2,\alpha}(x)| \leq C \frac{\mu_\alpha^2 \ln \left( 2 + \frac{d_g(x_\alpha, x)^2}{\mu_\alpha^2} \right)}{\mu_\alpha^2 + d_g(x_\alpha, x)^2}. \quad (4.37)$$

Independently, by the maximum principle, the  $w_{1,\alpha}$ 's satisfy that  $0 \leq w_{1,\alpha} \leq \frac{1}{q}$ . Let  $\hat{g}_\alpha(x) = (\exp_{x_\alpha}^* g)(\hat{r}_\alpha x)$  and  $\hat{w}_{1,\alpha}(x) = w_{1,\alpha}(\exp_{x_\alpha}(\hat{r}_\alpha x))$ . There holds

$$\begin{cases} \Delta_{\hat{g}_\alpha} \hat{w}_{1,\alpha} + m_1^2 \hat{r}_\alpha^2 \hat{w}_{1,\alpha} = 0 & \text{in } B \\ w_{1,\alpha} = \hat{v}_\alpha & \text{on } \partial B, \end{cases} \quad (4.38)$$

where  $B = B_0(1) \subset \mathbb{R}^4$ , and  $\hat{v}_\alpha(x) = v_\alpha(\exp_{x_\alpha}(\hat{r}_\alpha x))$ . At this point we claim that

$$r_\alpha \rightarrow 0 \quad (4.39)$$

as  $\alpha \rightarrow +\infty$ . In order to prove (4.39) we proceed by contradiction and assume that  $r_\alpha \geq \delta_0 > 0$  for all  $\alpha \gg 1$ . By Lemma 4.1 and (4.25),

$$v_\alpha \leq C \mu_\alpha^\beta \text{ in } M \setminus B_{x_\alpha}(r_\alpha), \quad (4.40)$$

where  $C > 0$  is independent of  $\alpha$  since we assumed  $r_\alpha \geq \delta_0 > 0$ . In particular,  $\|v_\alpha\|_{L^\infty(\partial B_\alpha)} \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Then  $\|\hat{v}_\alpha\|_{L^\infty(\partial B)} \rightarrow 0$  as  $\alpha \rightarrow +\infty$ , and it follows from the maximum principle and (4.38) that  $\|\hat{w}_{1,\alpha}\|_{L^\infty(B)} \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . In particular,  $\|w_{1,\alpha}\|_{L^\infty(B_\alpha)} \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . By (4.32) and (4.37), thanks to what we just obtained about the  $w_{1,\alpha}$ 's, we get that  $\|v_\alpha\|_{L^\infty(B_\alpha)} \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Then, by Lemma 4.1 and (4.29) we get that

$$\begin{aligned} \int_{B_{x_\alpha}(r_\alpha)} v_\alpha u_\alpha^2 dv_g &= o\left(\mu_\alpha^2 \ln \frac{1}{\mu_\alpha}\right) \\ \int_{B_{x_\alpha}(r_\alpha)} u_\alpha v_\alpha |X_\alpha(\nabla u_\alpha)| dv_g &= o\left(\mu_\alpha^2 \ln \frac{1}{\mu_\alpha}\right). \end{aligned} \quad (4.41)$$

and by (4.30) and (4.31), we obtain a contradiction with (0.7). This proves (4.39). By (4.18), (4.19) and (4.20) we get with (4.39) that

$$Q_{1,\alpha} = - (128\omega_3 \mathcal{H}(0) + o(1)) \mu_\alpha^2 r_\alpha^{-2}. \quad (4.42)$$

Now we distinguish the two cases:

- (i)  $r_\alpha^2 \ln \frac{r_\alpha}{\mu_\alpha} \rightarrow 0$  as  $\alpha \rightarrow +\infty$ , and
- (ii)  $r_\alpha^2 \ln \frac{r_\alpha}{\mu_\alpha} \geq \delta_0 > 0$  for all  $\alpha$ .

In case (i), since  $v_\alpha = O(1)$ , we get from Lemma 4.1 and (4.29) that

$$\begin{aligned} \int_{B_{x_\alpha}(r_\alpha)} v_\alpha u_\alpha^2 dv_g &= O\left(\mu_\alpha^2 \ln \frac{r_\alpha}{\mu_\alpha}\right), \\ \int_{B_{x_\alpha}(r_\alpha)} v_\alpha u_\alpha |X_\alpha(\nabla u_\alpha)| dv_g &= O\left(\mu_\alpha^2 \ln \frac{r_\alpha}{\mu_\alpha}\right). \end{aligned} \quad (4.43)$$

Since there also holds that  $r_\alpha^2 \ln \frac{1}{\mu_\alpha} \rightarrow +\infty$  it follows from (4.30), (4.42) and (4.43) that  $\mathcal{H}(0) = 0$ . Now we assume (ii). From (ii) we get that  $r_\alpha \geq C(\ln \frac{1}{\mu_\alpha})^{-1/2}$  and by (4.25) we obtain that

$$v_\alpha \leq C \left( \mu_\alpha \ln \frac{1}{\mu_\alpha} \right)^\beta \quad \text{in } M \setminus B_{x_\alpha}(r_\alpha).$$

In particular,  $\|v_\alpha\|_{L^\infty(\partial B_\alpha)} \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Then  $\|\hat{v}_\alpha\|_{L^\infty(\partial B)} \rightarrow 0$  as  $\alpha \rightarrow +\infty$ , and it follows from the maximum principle and (4.38) that  $\|\hat{w}_{1,\alpha}\|_{L^\infty(B)} \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . In particular,  $\|w_{1,\alpha}\|_{L^\infty(B_\alpha)} \rightarrow 0$  as  $\alpha \rightarrow +\infty$  and we get with (4.32), Lemma 4.1, and (4.29), that

$$\begin{aligned} \int_{B_{x_\alpha}(r_\alpha)} v_\alpha u_\alpha^2 dv_g &= \int_{B_{x_\alpha}(r_\alpha)} w_{2,\alpha} u_\alpha^2 dv_g + o\left(\mu_\alpha^2 \ln \frac{r_\alpha}{\mu_\alpha}\right), \\ \int_{B_{x_\alpha}(r_\alpha)} v_\alpha u_\alpha |X_\alpha(\nabla u_\alpha)| dv_g &= \\ \int_{B_{x_\alpha}(r_\alpha)} w_{2,\alpha} u_\alpha |X_\alpha(\nabla u_\alpha)| dv_g &+ o\left(\mu_\alpha^2 \ln \frac{r_\alpha}{\mu_\alpha}\right). \end{aligned} \quad (4.44)$$

There holds,

$$\Delta_g w_{2,\alpha} + m_1^2 w_{2,\alpha} = q(1 - qv_\alpha) u_\alpha^2. \quad (4.45)$$

Let  $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$  be such that  $\eta$  is smooth,  $0 \leq \eta \leq 1$ ,  $\eta = 1$  in  $B_0(1)$ , and  $\eta = 0$  in  $\mathbb{R}^n \setminus B_0(2)$ . We define

$$\eta_\alpha(x) = \eta\left(\frac{d_g(x_\alpha, x)}{r_\alpha}\right) \quad (4.46)$$

so that  $\eta_\alpha = 1$  in  $B_{x_\alpha}(r_\alpha)$  and  $\eta_\alpha = 0$  in  $M \setminus B_{x_\alpha}(2r_\alpha)$ . By Hölder's inequalities,

$$\begin{aligned} \int_{B_{x_\alpha}(r_\alpha)} w_{2,\alpha} u_\alpha^2 dv_g &\leq \left( \int_{B_{x_\alpha}(r_\alpha)} w_{2,\alpha}^4 dv_g \right)^{1/4} \left( \int_{B_{x_\alpha}(r_\alpha)} u_\alpha^{8/3} dv_g \right)^{3/4} \quad \text{and} \\ \int_{B_{x_\alpha}(r_\alpha)} u_\alpha w_{2,\alpha} |X_\alpha(\nabla u_\alpha)| dv_g & \\ \leq \left( \int_{B_{x_\alpha}(r_\alpha)} w_{2,\alpha}^4 dv_g \right)^{1/4} \left( \int_{B_{x_\alpha}(r_\alpha)} |u_\alpha X_\alpha(\nabla u_\alpha)|^{4/3} dv_g \right)^{3/4}, \end{aligned} \quad (4.47)$$

while by Lemma 4.1 and (4.29) there holds that

$$\begin{aligned} \int_{B_{x_\alpha}(r_\alpha)} u_\alpha^{8/3} dv_g &= O\left(\mu_\alpha^{4/3}\right) \quad \text{and} \\ \int_{B_{x_\alpha}(r_\alpha)} |u_\alpha X_\alpha(\nabla u_\alpha)|^{4/3} dv_g &= O\left(\mu_\alpha^{4/3}\right). \end{aligned} \quad (4.48)$$

Multiplying (4.45) by  $\eta_\alpha^2 w_{2,\alpha}$ , and integrating over  $M$ , we get that

$$\int_M (\Delta_g w_{2,\alpha} + m_1^2 w_{2,\alpha}) \eta_\alpha^2 w_{2,\alpha} dv_g \leq q \int_M u_\alpha^2 w_{2,\alpha} \eta_\alpha^2 dv_g . \quad (4.49)$$

By Hölder's and Sobolev inequalities, and by (4.48),

$$\int_M u_\alpha^2 w_{2,\alpha} \eta_\alpha^2 dv_g \leq C \mu_\alpha \|\eta_\alpha w_{2,\alpha}\|_{H^1} \quad (4.50)$$

and it follows from (4.49) and (4.50) that

$$\|\eta_\alpha w_{2,\alpha}\|_{H^1}^2 \leq \int_M |\nabla \eta_\alpha|^2 w_{2,\alpha}^2 dv_g + C \mu_\alpha \|\eta_\alpha w_{2,\alpha}\|_{H^1} . \quad (4.51)$$

By (4.37), since  $|\nabla \eta_\alpha| \leq C r_\alpha^{-1}$ , we get that

$$\int_M |\nabla \eta_\alpha|^2 w_{2,\alpha}^2 dv_g \leq C r_\alpha^2 \left( \frac{\mu_\alpha^2}{r_\alpha^2} \ln \left( \frac{r_\alpha}{\mu_\alpha} \right) \right)^2$$

and by (4.9) it follows that

$$\int_M |\nabla \eta_\alpha|^2 w_{2,\alpha}^2 dv_g = o \left( \mu_\alpha^2 \ln^2 \frac{1}{\mu_\alpha} \right) .$$

Coming back to (4.51), it follows that

$$\|\eta_\alpha w_{2,\alpha}\|_{H^1} = o \left( \mu_\alpha \ln \frac{1}{\mu_\alpha} \right) . \quad (4.52)$$

By (4.44), (4.47) and (4.48), we then get with (4.52) that

$$\begin{aligned} \int_{B_{x_\alpha}(r_\alpha)} v_\alpha u_\alpha^2 dv_g &= o \left( \mu_\alpha^2 \ln \frac{1}{\mu_\alpha} \right) \quad \text{and} \\ \int_{B_{x_\alpha}(r_\alpha)} u_\alpha v_\alpha |X_\alpha(\nabla u_\alpha)| dv_g &= o \left( \mu_\alpha^2 \ln \frac{1}{\mu_\alpha} \right) . \end{aligned} \quad (4.53)$$

Coming back to (4.30) and (4.42) it follows that

$$\mathcal{H}(0) = \frac{1}{64} \left( m_0^2 - \omega^2 - \frac{1}{6} S_g(x_0) \right) \lim_{\alpha \rightarrow +\infty} r_\alpha^2 \ln \frac{r_\alpha}{\mu_\alpha} . \quad (4.54)$$

By (4.54) we get that  $\mathcal{H}(0) \leq 0$ . At this point it remains to prove that  $\rho_\alpha = O(r_\alpha)$ . We prove that  $\rho_\alpha = r_\alpha$ . If not the case, then  $r_\alpha < \rho_\alpha$  and we get with (4.11) that  $(r\varphi(r))'(1) = 0$ , where

$$\begin{aligned} \varphi(r) &= \frac{1}{\omega_3 r^3} \int_{\partial B_0(r)} \tilde{u} d\sigma \\ &= \frac{8}{r^2} + \mathcal{H}(0) . \end{aligned}$$

Hence  $\mathcal{H}(0) = 8$  and we get a contradiction with  $\mathcal{H}(0) \leq 0$ . In other words,  $\rho_\alpha = r_\alpha$  for all  $\alpha \gg 1$ . This ends the proof of the lemma.  $\square$

Thanks to Lemma 4.2 we can now prove the uniform bounds in Theorem 0.3. This is the subject of what follows.

*Proof of the uniform bounds in Theorem 0.3.* Let  $(M, g)$  be a smooth compact Riemannian 4-dimensional manifold and  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of (4.1) such that (0.7) holds true. By Druet, Hebey and V  tois [22] there exists  $C > 0$  such that for any  $\alpha$  the following holds true: there exist  $N_\alpha \in \mathbb{N}^*$  and  $N_\alpha$  critical points of  $u_\alpha$ , denoted by  $(x_{1,\alpha}, x_{2,\alpha}, \dots, x_{N_\alpha,\alpha})$ , such that

$$d_g(x_{i,\alpha}, x_{j,\alpha}) u_\alpha(x_{i,\alpha}) \geq 1 \quad (4.55)$$

for all  $i, j \in \{1, \dots, N_\alpha\}$ ,  $i \neq j$ , and

$$\left( \min_{i=1, \dots, N_\alpha} d_g(x_{i,\alpha}, x) \right) u_\alpha(x) \leq C \quad (4.56)$$

for all  $x \in M$  and all  $\alpha$ . We define

$$d_\alpha = \min_{1 \leq i < j \leq N_\alpha} d_g(x_{i,\alpha}, x_{j,\alpha}) . \quad (4.57)$$

If  $N_\alpha = 1$ , we set  $d_\alpha = \frac{1}{4}i_g$ , where  $i_g$  is the injectivity radius of  $(M, g)$ . We claim that

$$d_\alpha \not\rightarrow 0 \quad (4.58)$$

as  $\alpha \rightarrow +\infty$ . In order to prove this claim, we proceed by contradiction. Assuming on the contrary that  $d_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ , we see that  $N_\alpha \geq 2$  for  $\alpha$  large, and we can thus assume that the concentration points are ordered in such a way that

$$d_\alpha = d_g(x_{1,\alpha}, x_{2,\alpha}) \leq d_g(x_{1,\alpha}, x_{3,\alpha}) \leq \dots \leq d_g(x_{1,\alpha}, x_{N_\alpha,\alpha}) . \quad (4.59)$$

We set, for  $x \in B_0(\delta d_\alpha^{-1})$ ,  $0 < \delta < \frac{1}{2}i_g$  fixed,

$$\begin{aligned} \hat{u}_\alpha(x) &= d_\alpha u_\alpha \left( \exp_{x_{1,\alpha}}(d_\alpha x) \right) , \\ \hat{h}_\alpha(x) &= h_\alpha \left( \exp_{x_{1,\alpha}}(d_\alpha x) \right) , \text{ and} \\ \hat{g}_\alpha(x) &= \left( \exp_{x_{1,\alpha}}^* g \right) (d_\alpha x) . \end{aligned}$$

It is clear that  $\hat{g}_\alpha \rightarrow \xi$  in  $C_{loc}^2(\mathbb{R}^n)$  as  $\alpha \rightarrow +\infty$  since  $d_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Thanks to (4.1) we have that

$$\Delta_{\hat{g}_\alpha} \hat{u}_\alpha + d_\alpha^2 \hat{h}_\alpha \hat{u}_\alpha = \hat{u}_\alpha^3 \quad (4.60)$$

in  $B_0(\delta d_\alpha^{-1})$ , for all  $i$ . For any  $R > 0$ , we also let  $1 \leq N_{R,\alpha} \leq N_\alpha$  be such that

$$\begin{aligned} d_g(x_{1,\alpha}, x_{i,\alpha}) &\leq R d_\alpha \text{ for } 1 \leq i \leq N_{R,\alpha} , \text{ and} \\ d_g(x_{1,\alpha}, x_{i,\alpha}) &> R d_\alpha \text{ for } N_{R,\alpha} + 1 \leq i \leq N_\alpha . \end{aligned}$$

Such a  $N_{R,\alpha}$  does exist thanks to (4.59). We also have that  $N_{R,\alpha} \geq 2$  for all  $R > 1$  and that  $(N_{R,\alpha})_\alpha$  is uniformly bounded for all  $R > 0$  thanks to (4.57). In the sequel, we set

$$\hat{x}_{i,\alpha} = d_\alpha^{-1} \exp_{x_{1,\alpha}}^{-1}(x_{i,\alpha})$$

for all  $1 \leq i \leq N_\alpha$  such that  $d_g(x_{1,\alpha}, x_{i,\alpha}) \leq \frac{1}{2}i_g$ . Thanks to (4.56), for any  $R > 1$ , there exists  $C_R > 0$  such that

$$\sup_{B_0(R) \setminus \bigcup_{i=1}^{N_{2R,\alpha}} B_{\hat{x}_{i,\alpha}}(\frac{1}{R})} \hat{u}_\alpha \leq C_R . \quad (4.61)$$

By the Harnack inequality in Druet, Hebey and Vétois [22], for any  $R > 1$ , there exists  $D_R > 1$  such that

$$\|\nabla \hat{u}_\alpha\|_{L^\infty(\Omega_{R,\alpha})} \leq D_R \sup_{\Omega_{R,\alpha}} \hat{u}_\alpha \leq D_R^2 \inf_{\Omega_{R,\alpha}} \hat{u}_\alpha, \quad (4.62)$$

where

$$\Omega_{R,\alpha} = B_0(R) \setminus \bigcup_{i=1}^{N_{2R,\alpha}} B_{\hat{x}_{i,\alpha}}\left(\frac{1}{R}\right).$$

Assume first that, for some  $R > 0$ , there exists  $1 \leq i \leq N_{R,\alpha}$  such that

$$\hat{u}_\alpha(\hat{x}_{i,\alpha}) = O(1). \quad (4.63)$$

The two first equations in (4.4) are satisfied by the sequences  $x_\alpha = x_{i,\alpha}$  and  $\rho_\alpha = \frac{1}{8}d_\alpha$ . Then it follows from (4.6) that the last equation in (4.4) cannot hold and thus that  $(\hat{u}_\alpha)_\alpha$  is uniformly bounded in  $B_{\hat{x}_{i,\alpha}}(\frac{3}{4})$ . In particular, by standard elliptic theory, and thanks to (4.60),  $(\hat{u}_\alpha)_\alpha$  is uniformly bounded in  $C^1(B_{\hat{x}_{i,\alpha}}(\frac{1}{2}))$ . Since, by (4.55), we have that

$$|\hat{x}_{i,\alpha}|^{\frac{n-2}{2}} |\hat{u}_\alpha(\hat{x}_{i,\alpha})| \geq 1,$$

we get the existence of some  $\delta_i > 0$  such that

$$|\hat{u}_\alpha| \geq \frac{1}{2} |\hat{x}_{i,\alpha}|^{1-\frac{n}{2}} \geq \frac{1}{2} R^{1-\frac{n}{2}}$$

in  $B_{\hat{x}_{i,\alpha}}(\delta_i)$ . Assume now that, for some  $R > 0$ , there exists  $1 \leq i \leq N_{R,\alpha}$  such that

$$|\hat{u}_\alpha(\hat{x}_{i,\alpha})| \rightarrow +\infty \quad (4.64)$$

as  $\alpha \rightarrow +\infty$ . Since (4.4) is satisfied by the sequences  $x_\alpha = x_{i,\alpha}$  and  $\rho_\alpha = \frac{1}{8}d_\alpha$ , it follows from Lemma 4.2 that the sequence  $(|\hat{u}_\alpha(\hat{x}_{i,\alpha})| \times |\hat{u}_\alpha|)_\alpha$  is uniformly bounded in

$$\hat{\Omega}_\alpha = B_{\hat{x}_{i,\alpha}}(\tilde{\delta}_i) \setminus B_{\hat{x}_{i,\alpha}}(\frac{\tilde{\delta}_i}{2})$$

for some  $\tilde{\delta}_i > 0$ . Thus, using (4.62), we can deduce that these two situations are mutually exclusive in the sense that either (4.63) holds true for all  $i$  or (4.64) holds true for all  $i$ . Now we split the conclusion of the proof into two cases.

In the first case we assume that there exist  $R > 0$  and  $1 \leq i \leq N_{R,\alpha}$  such that  $\hat{u}_\alpha(\hat{x}_{i,\alpha}) = O(1)$ . Then, thanks to the above discussion, we get that  $\hat{u}_\alpha(\hat{x}_{j,\alpha}) = O(1)$  for all  $1 \leq j \leq N_{R,\alpha}$  and all  $R > 0$ . As above, we get that  $(\hat{u}_\alpha)_\alpha$  is uniformly bounded in  $C_{loc}^1(\mathbb{R}^4)$ . Thus, by standard elliptic theory, there exists a subsequence of  $(\hat{u}_\alpha)_\alpha$  which converges in  $C_{loc}^1(\mathbb{R}^4)$  to some  $\hat{u}$  solution of  $\Delta \hat{u} = \hat{u}^3$  in  $\mathbb{R}^4$ . By the above discussion,  $|u|$  possesses at least two critical points, namely 0 and  $\hat{x}_2$ , the limit of  $\hat{x}_{2,\alpha}$ . This is absurd thanks to the classification of Caffarelli, Gidas and Spruck [14].

In the second case we assume that there exist  $R > 0$  and  $1 \leq i \leq N_{R,\alpha}$  such that  $|\hat{u}_\alpha(\hat{x}_{i,\alpha})| \rightarrow +\infty$  as  $\alpha \rightarrow +\infty$ . Then, thanks to the above discussion,  $\hat{u}_\alpha(\hat{x}_{j,\alpha}) \rightarrow +\infty$  as  $\alpha \rightarrow +\infty$ , for all  $1 \leq j \leq N_{R,\alpha}$  and all  $R > 0$ . By (4.60) we have that

$$\Delta_{\hat{g}_\alpha} \hat{v}_\alpha + d_\alpha^2 \hat{h}_\alpha \hat{v}_\alpha = \frac{1}{\hat{u}_\alpha(0)^2} \hat{v}_\alpha^3,$$

where  $\hat{v}_\alpha = \hat{u}_\alpha(0)\hat{u}_\alpha$ . Applying Lemma 4.2 and standard elliptic theory, and thanks to (4.62) and to the above discussion, one easily checks that, after passing to a subsequence,  $\hat{u}_\alpha(0)\hat{u}_\alpha \rightarrow \hat{G}$  in  $C_{loc}^1(\mathbb{R}^n \setminus \{\hat{x}_i\}_{i \in I})$  as  $\alpha \rightarrow +\infty$ , where  $I = \{1, \dots, \lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} N_{R,\alpha}\}$  and, for any  $R > 0$ ,

$$\hat{G}(x) = \sum_{i=1}^{\tilde{N}_R} \frac{\Lambda_i}{|x - \hat{x}_i|^2} + \hat{H}_R(x)$$

in  $B_0(R)$ , where  $2 \leq \tilde{N}_R \leq N_{2R}$  is such that  $|\hat{x}_{\tilde{N}_R}| \leq R$  and  $|\hat{x}_{\tilde{N}_R+1}| > R$ , where  $N_{2R,\alpha} \rightarrow N_{2R}$  as  $\alpha \rightarrow +\infty$ , where  $\lambda_i > 0$ , and where  $\hat{H}_R$  is a harmonic function in  $B_0(R)$ . Since  $\hat{G} \geq 0$ , we can write thanks to the maximum principle that, in a neighbourhood of the origin,

$$\hat{G}(x) = \frac{\Lambda_1}{|x|^{n-2}} + \hat{H}(x),$$

where  $\hat{H}(0) \geq \Lambda_2 - \Lambda_1 R^{-2} - \Lambda_2(R-1)^{-2}$ . Choosing  $R$  large enough, we can ensure that  $\hat{H}(0) > 0$  and this is in contradiction with Lemma 4.2.

By the above discussion we get that (4.58) holds true. Clearly, this implies that  $(N_\alpha)_\alpha$  is uniformly bounded. Let  $(x_\alpha)_\alpha$  be a sequence of maximal points of  $u_\alpha$ . Thanks to (4.3) and to (4.58), we clearly have that (4.4) holds true for the sequences  $(x_\alpha)_\alpha$  and  $\rho_\alpha = \delta$  for some  $\delta > 0$  fixed. This clearly contradicts Lemma 4.2 and thus concludes the proof of the uniform bounds in Theorem 0.3.  $\square$

Existence and nonexistence of a priori estimates for critical elliptic Schrödinger type equations on manifolds have been investigated by Berti-Malchiodi [7], Brendle [8, 9], Brendle and Marques [10], Brézis and Li [11], Druet [15, 16], Druet and Hebey [17, 18, 19], Druet, Hebey, and Vétoris [22], Druet and Laurain [23], Hebey [27, 28], Khuri, Marques and Schoen [30], Li and Zhang [32, 33], Li and Zhu [34], Marques [39], Micheletti, Pistoia and Vétoris [40], Schoen [44, 45], and Vétoris [48]. In the subcritical case, a priori estimates for Schrödinger equations go back to the seminal work by Gidas and Spruck [24]. The above list is not exhaustive.

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E. HEBEY, UNIVERSITÉ DE CERGY-PONTOISE, DÉPARTEMENT DE MATHÉMATIQUES, SITE DE SAINT-MARTIN, 2 AVENUE ADOLPHE CHAUVIN, 95302 CERGY-PONTOISE CEDEX, FRANCE  
*E-mail address:* `Emmanuel.Hebey@math.u-cergy.fr`

T.T. TRUONG, UNIVERSITÉ DE CERGY-PONTOISE, DÉPARTEMENT DE PHYSIQUE, SITE DE SAINT-MARTIN, 2 AVENUE ADOLPHE CHAUVIN, 95302 CERGY-PONTOISE CEDEX, FRANCE  
*E-mail address:* `tuong.truong@u-cergy.fr`